First-Order homogeneous differential equation

Given the following differential equation

$$x\frac{dy}{dx} - y = xe^{\frac{y}{x}}$$

1. Determine the general solution of the differential equation.
2. Determine the particular solution of the differential equation if $y(1) = 3$.

Solution

1.

$$y'(x) = \frac{dy}{dx}$$

$$x\frac{dy}{dx} - y = xe^{\frac{y}{x}}$$

$$xdy - ydx = xe^{\frac{y}{x}}dx$$

$$xdy = \left(xe^{\frac{y}{x}} + y\right)dx$$

We see that on both sides we have a homogeneous function of degree 1. So we set $u = \frac{y}{x}$, then $y = ux$ and $dy = udx + xdu$. Also, we divide both sides by x^n where n is the degree of homogeneity.

$$(udx + xdu) = (e^u + u)\,dx$$

$$udx + xdu = e^u dx + udx$$

$$xdu = e^u dx$$

$$\frac{du}{e^u} = \frac{dx}{x}$$

Integrate both sides of the equation:

$$\int \frac{1}{e^u}du = \int \frac{1}{x}dx$$

$$-\frac{1}{e^u} = \ln(x) + C$$

$$\frac{1}{e^u} = C - \ln(x)$$

Substitute back:

$$u = \frac{y}{x}$$

$$-\frac{1}{e^{\frac{y}{x}}} = \ln(x) + C$$

Solve for y:

$$K - \ln(x) = \frac{1}{e^{\frac{y}{x}}}$$

$$K - \ln(x) = e^{-\frac{y}{x}}$$

$$\ln(K - \ln(x)) = -\frac{y}{x}$$

$$y = -x\ln(K - \ln(x))$$

2. Substituting the initial conditions into the previous solution:

$$y = -x\ln(C - \ln(x))$$

At $x = 1$, $y = 3$

$$3 = -\ln(K)$$

$$K = \frac{1}{e^3}$$

Substituting the found coefficients into the general solution:

$$y = -x\ln\left(\frac{1}{e^3} - \ln(x)\right)$$